How to impose a Neumann boundary condition with the lattice Boltzmann method

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1 The von Neumann boundary condition with the lattice Boltzmann method

It is now quite well known how to impose a Dirichlet boundary condition (BC) on a flat wall with the lattice Boltzmann method (LBM) (see among others [3], [1], [2]). The question of the von Neumann BC is more tricky since we usually do not have direct access on gradients with the LBM. Therefore it might be difficult to apply an zero velocity gradient at the outlet of a simulation or an adiabatic wall for thermal simulations. Fortunately there is an easy way to deal with these kind of problems and the method will be described here.

To clarify the notations say that we are resolving an advection-diffusion equation in two dimensions and that we want to impose an adiabatic BC :

$$\boldsymbol{n}(\boldsymbol{\nabla}T)|_{\partial\Omega} = 0, \tag{1}$$

where T is the temperature, $\boldsymbol{n}$ is the normal vector pointing outwards the computational domain Ω and $\partial\Omega$ is the boundary.

Without loss of generality we will assume that $\boldsymbol{n} = (-1, 0)$. Therefore Eq. (1) reads

$$\partial_x T|_{\partial\Omega} = 0. \tag{2}$$

Let the adiabatic wall be situated at position $x = x_0$ for all y (we will ignore the y coordinate in what follows). The value of T for $x > x_0$ is known since it is in the bulk. The idea now is to use a Dirichlet BC that will embody the von Neumann BC. To do so let us do a Taylor expansion of T around x_0 up to second order (the LBM is second order accurate therefore it seems natural to take second order Taylor expansion) :

$$T(x_0+\delta x) = T(x_0) + \delta x T'(x_0) + \frac{\delta x^2}{2} T''(x_0) + \mathcal{O}(\delta x^3), \tag{3}$$

$$T(x_0+2\delta x) = T(x_0) + 2\delta x T'(x_0) + 2\delta x^2 T''(x_0) + \mathcal{O}(\delta x^3). \tag{4}$$

Taking $\delta x = 1$ the spacing between two lattice sites, we know the values of $T(x_0+1)$ and $T(x_0+2)$. Furthermore we want to impose $T'(x_0) = 0$. Therefore we are left with two equations and two unknowns. We easily solve the system for $T(x_0)$ and get

$$T(x_0) = \frac{4T(x_0+1) - T(x_0+2)}{3}. \tag{5}$$

Now imposing this temperature as a *Dirichlet* BC one automatically satisfies the von Neumann BC. Of course this BC is not local since you need information about two nearest neighbors (for second order expansion). For first order expansion one has

$$T(x_0) = T(x_0+1). \tag{6}$$

References

[1] T. Inamuro, M. Yoshina, and F. Ogino. A non-slip boundary condition for lattice Boltzmann simulations. *Phys. Fluids*, 7:2928–2930, 1995.

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[3] Q. Zou and X. He. On pressure and velocity boundary conditions for the lattice Boltzmann BGK model. *Phys. Fluids*, 9:1592–1598, 1997.